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Two-loop QCD gauge coupling at high temperatures

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Abstract: We determine the 2-loop effective gauge coupling of QCD at high temperatures, defined as a matching coefficient appearing in the dimensionally reduced effective field theory. The result allows to improve on one of the classic non-perturbative probes for the convergence of the weak-coupling expansion at high temperatures, the comparison of full and effective theory determinations of an observable called the spatial string tension. We find surprisingly good agreement almost down to the critical temperature of the deconfinement phase transition. We also determine one new contribution of order $\mathcal{O}(g^6T^4)$ to the pressure of hot QCD.

Keywords: Lattice Gauge Field Theories, Thermal Field Theory, QCD, NLO Computations.

Contents

1. Introduction

Indirect signs for rapid thermalisation after heavy ion collisions at RHIC energies, derived for instance from the fact that hydrodynamic models assuming local thermodynamic equilibrium appear to work very well [1], have underlined the need to understand the physics of thermal QCD at temperatures above a few hundred MeV.

Given asymptotic freedom, a natural tool for these studies is the weak-coupling expansion [2]. Alas, it has been known since a long time that the weak-coupling expansion converges very slowly at all realistic temperatures [3, 4]. It also has theoretically a non-trivial structure, with odd powers of the gauge coupling [5] and even coefficients that can only be determined non-perturbatively [6, 7].

On the other hand, the degrees of freedom responsible for the slow convergence can be identified [8 − 10]: they are the "soft" static colour-electric modes, parametrically $p \sim gT$ (leading to the odd powers in the gauge coupling), as well as the "ultrasoft" static colour-magnetic modes, parametrically $p \sim g^2T$ (leading to the non-perturbative coefficients in the weak-coupling expansion). Here p denotes the characteristic momentum scale, g the gauge coupling and T the temperature. The belief has been that perturbation theory restricted to parametrically hard scales $p \sim 2\pi T$ alone should converge well, while the soft and the ultrasoft scales need to be treated either with "improved" analytic schemes, or then non-perturbatively. As a starting point for these demanding tasks one may take, however, either the dimensionally reduced effective field theory [11, 12] or the hard thermal loop effective theory [13], which have been obtained by integrating out the parametrically hard scales.

Quantitative evidence for this picture can be obtained by choosing simple observables which can be determined reliably both with four-dimensional (4d) lattice simulations and with the soft/ultrasoft effective theory. This forces us to restrict to static observables and, for the moment, mostly pure gauge theory. Various comparisons of this kind are summarised in references [14 – 16]. The most precise results are related to static correlation lengths in various quantum number channels [17], where good agreement has generally been found down to $T \sim 2T_c$, where T_c is the critical temperature of the deconfinement phase transition. The thermodynamic pressure of QCD is also consistent with this picture [10], even though that comparison is not unambiguous yet, due to the fact that the effective theory approach does not directly produce the physical number, but requires not-yet-determined ultraviolet matching coefficients for its interpretation [18].[1]

The purpose of this paper is to study another observable for which an unambiguous comparison is possible. The observable is the "spatial string tension", σ_s. 4d lattice determinations of σ_s in pure SU(3) gauge theory exist since a while already [22] but, as has most recently been stressed in reference [23], the comparison with effective theory results shows a clear discrepancy. In order to improve on the resolution on the effective theory side, we compute here the gauge coupling of the dimensionally reduced theory up to 2-loop order. Combining with other ingredients [24, 25], to be specified below, allows then for a precise comparison. We find that once the 2-loop corrections are included, the match to 4d lattice data improves quite significantly and supports the picture outlined above.

The plan of this paper is the following. In section 2 we present the 2-loop computation of the effective gauge coupling of the dimensionally reduced theory. In section 3 we discuss the numerical evaluation of this result. In section 4 we use the outcome for estimating the spatial string tension, and compare with 4d lattice data. We conclude in section 5.

2. Effective gauge coupling

We consider finite temperature QCD with the gauge group SU(N_c), and N_f flavours of massless quarks. In dimensional regularisation the bare euclidean lagrangian reads, before gauge fixing,

$$S_{\rm QCD} = \int_0^\beta {\rm d}\tau \int {\rm d}^d x\, \mathcal{L}_{\rm QCD}\,, \tag{2.1}$$

$$\mathcal{L}_{\rm QCD} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar\psi \gamma_\mu D_\mu \psi\,, \tag{2.2}$$

where $\beta = T^{-1}$, $d = 3 - 2\epsilon$, $\mu, \nu = 0, \ldots, d$, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_B f^{abc} A^b_\mu A^c_\nu$, $D_\mu = \partial_\mu - i g_B A_\mu$, $A_\mu = A^a_\mu T^a$, T^a are hermitean generators of SU(N_c) normalised such that ${\rm Tr}\,[T^a T^b] = \delta^{ab}/2$, $\gamma^\dagger_\mu = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, g_B is the bare gauge coupling, and ψ carries Dirac, colour, and flavour indices. We use the standard symbols $C_A = N_c, C_F = (N_c^2 - 1)/(2N_c), T_F = N_f/2$ for the various group theory factors emerging.

[1] For the status regarding a few other observables, see references [19 – 21].

At high enough temperatures, the dynamics of eq. (2.2) is contained in a simpler, dimensionally reduced effective field theory [11, 12, 8]:

$$S_{\rm EQCD} = \int {\rm d}^d x \mathcal{L}_{\rm EQCD} \,, \tag{2.3}$$

$$\mathcal{L}_{\rm EQCD} = \frac{1}{4} F^a_{ij} F^a_{ij} + {\rm Tr}\,[D_i, B_0]^2 + m_{\rm E}^2 {\rm Tr}\,[B_0^2] + \lambda_{\rm E}^{(1)} ({\rm Tr}\,[B_0^2])^2 + \lambda_{\rm E}^{(2)} {\rm Tr}\,[B_0^4] + \cdots \,. \tag{2.4}$$

Here $i = 1, \ldots, d$, $F_{ij} = \partial_i B^a_j - \partial_j B^a_i + g_{\rm E} f^{abc} B^b_i B^c_j$, and $D_i = \partial_i - i g_{\rm E} B_i$. The fields B^a_μ have the dimension $[{\rm GeV}]^{1/2-\epsilon}$, due to a trivial rescaling with $T^{1/2}$. Note also that the quartic couplings $\lambda_{\rm E}^{(1)}$, $\lambda_{\rm E}^{(2)}$ are linearly dependent for $N_c \leq 3$, since then ${\rm Tr}\,[B_0^4] = \frac{1}{2}\,({\rm Tr}\,[B_0^2])^2$.

The theory in eq. (2.4) has been truncated to be super-renormalisable; that is, higher order operators [27] (see also references [28, 29] and references therein) have been dropped. The relative error thus induced has been discussed for generic Green's functions in reference [30], and for the particular case of the pressure of hot QCD in reference [10]. In the following we concentrate on an observable dynamically determined by the colour-magnetic scale $p \sim g^2 T$, and it is easy to see that in this case the higher order operators do not play any role at the order we are working.

The effective parameters in eq. (2.4) can be determined by matching, that is, by requiring that QCD and EQCD produce the same results, within the domain of validity of the latter theory. It is essential that infrared (IR) physics be treated in the same way in both theories at the matching stage and, as outlined in reference [8], the most convenient implementation of this requirement is to perform computations on both sides using "unresummed" propagators. We follow this procedure here.

The matching simplifies further by using the background field gauge (reference [31] and references therein). As this is essential for what follows, we start by briefly recalling the basic advantage of this approach. For a concise yet rigorous overview of the technique, see reference [32].

We denote the background gauge potential with B^a_μ, and the gauge-invariant combination following from $F^a_{\mu\nu}(B) F^a_{\mu\nu}(B)$ symbolically as $B^2 + g B^3 + g^2 B^4$. Now, the computation of the effective Lagrangian by integrating out the hard scales $p \sim 2\pi T$ produces, in general, an expression of the type

$$\mathcal{L}_{\rm eff} \sim c_2 B^2 + c_3\, g B^3 + c_4 g^2 B^4 + \cdots \,, \tag{2.5}$$

where c_i are coefficients of the form $c_i = 1 + \mathcal{O}(g^2)$. As the next step we are free to define a canonically normalised effective field $B_{\rm eff}$ as $B^2_{\rm eff} \equiv c_2 B^2$. Then the effective Lagrangian obtains the form

$$\mathcal{L}_{\rm eff} \sim B^2_{\rm eff} + c_3 c_2^{-3/2}\, g B^3_{\rm eff} + c_4 c_2^{-2}\, g^2 B^4_{\rm eff} + \cdots \,. \tag{2.6}$$

We can now read off the effective gauge coupling from the gauge-invariant structure:

$$g_{\rm eff} = c_3 c_2^{-3/2}\, g = c_4^{1/2} c_2^{-1} g \,. \tag{2.7}$$

We observe that two independent computations are needed for the determination of $g_{\rm eff}$, but we can choose whether to go through the 3-point or the 4-point function, in addition to the 2-point function (that is, using c_3 or c_4, in addition to c_2).

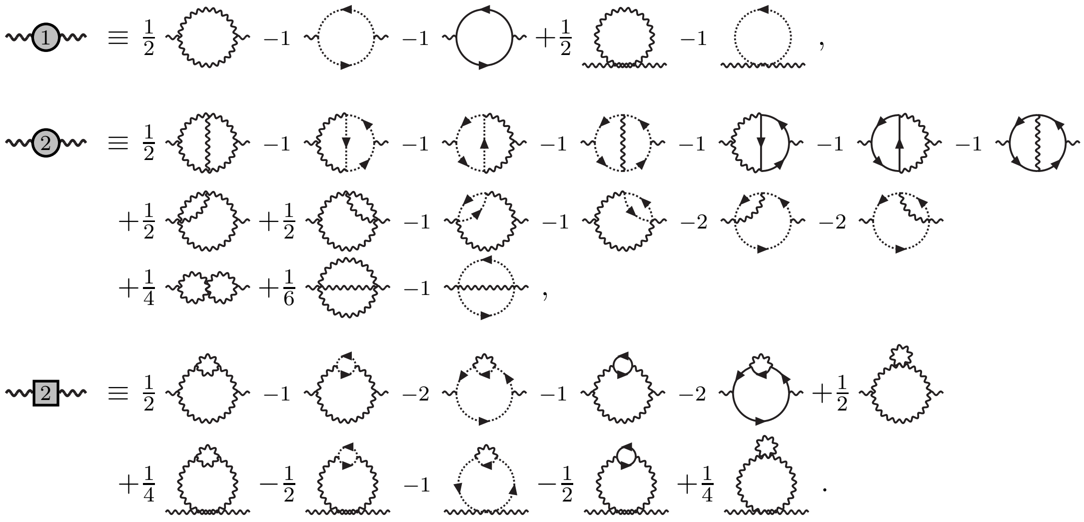

Figure 1: The 1-loop and 2-loop self-energy diagrams in the background field gauge. Wavy lines represent gauge fields, dotted lines ghosts, and solid lines fermions. The 2-loop graphs have been divided into two-particle-irreducible and two-particle-reducible contributions.

The background field gauge economises this setup. Indeed, the effective action is then gauge-invariant not only in terms of $B_{\rm eff}$, but also in terms of the original field B [31]. Writing equation (2.5) as

$$\mathcal{L}_{\rm eff} \sim c_2 \left[B^2 + c_3 c_2^{-1} g B^3 + c_4 c_2^{-1} g^2 B^4 \right] + \cdots , \tag{2.8}$$

gauge invariance in terms of B now tells us that $c_3 = c_2$ and $c_4 = c_2$. Combining with equation (2.7), we obtain

$$g_{\rm eff} = c_2^{-1/2} g \,, \tag{2.9}$$

so that it is enough to carry out one single 2-point computation, in order to obtain $g_{\rm eff}$. In our case, the role of $g_{\rm eff}$ is played by $g_{\rm E}$ (cf. equation (2.4)).

The class of background field gauges still allows for a general (bare) gauge parameter, ξ. As a cross-check we have carried out all computations with a general ξ, and verified that it cancels at the end. To be definite, we denote $(\xi)_{\rm here} = 1 - (\xi)_{\rm standard}$, so that the gauge field propagator reads

$$\Big\langle A^a_\mu(q) A^b_\nu(-q) \Big\rangle = \delta^{ab} \left[\frac{\delta_{\mu\nu}}{q^2} - \xi \frac{q_\mu q_\nu}{(q^2)^2} \right]. \tag{2.10}$$

In order to match the effective gauge coupling, we need to compute the 2-loop gluon self-energy, $\Pi_{\mu\nu}(p)$, for the background gauge potential B^a_μ. The graphs entering are shown in fig. 1. The external momentum p is taken purely spatial, $p = (0, \mathbf{p})$, while the heat bath is timelike, with euclidean four-velocity $u = (1, 0)$, so that $u \cdot u = 1, u \cdot p = 0$. In this case $\Pi_{\mu\nu}$ has three independent components (Π_{0i}, Π_{i0} vanish identically),

$$\Pi_{00}(\mathbf{p}) \equiv \Pi_{\rm E}(\mathbf{p}^2) \,, \qquad \Pi_{ij}(\mathbf{p}) \equiv \left(\delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2} \right) \Pi_{\rm T}(\mathbf{p}^2) + \frac{p_i p_j}{\mathbf{p}^2} \Pi_{\rm L}(\mathbf{p}^2) \,, \tag{2.11}$$

where $i, j = 1, \ldots, d$. In fact loop corrections to the spatially longitudinal part $\Pi_{\rm L}$ also vanish, so that only two non-trivial functions, $\Pi_{\rm E}, \Pi_{\rm T}$, remain.

Since we are carrying out a matching computation, any possible IR divergences cancel as we subtract the contribution of EQCD. Therefore we may Taylor-expand $\Pi_{\mu\nu}(p)$ to second order in $\mathbf{p}^2$. This leads to the nice simplification that the results on the EQCD side vanish identically in dimensional regularization, due to the absence of any mass scales in the propagators. Thus we only need to compute unresummed integrals on the QCD side.

After the Taylor-expansion, the 2-loop QCD integrals can all be cast in the form

$$I(i_1,i_2;j_1,j_2,j_3;k_1,k_2,k_3) \equiv \sum\!\!\!\!\!\!\!\int_{q,r} \frac{q_0^{i_1} r_0^{i_2} (\mathbf{q}\cdot\mathbf{p})^{j_1} (\mathbf{r}\cdot\mathbf{p})^{j_2} (\mathbf{q}\cdot\mathbf{r})^{j_3}}{[q_0^2+\mathbf{q}^2]^{k_1}[r_0^2+\mathbf{r}^2]^{k_2}[(q_0+r_0)^2+(\mathbf{q}+\mathbf{r})^2]^{k_3}} \,. \tag{2.12}$$

The indices here are non-negative integers, and the measure is the standard Matsubara sum-integral (bosonic or fermionic), with the spatial part $\int \mathrm{d}^d\mathbf{q}/(2\pi)^d \int \mathrm{d}^d\mathbf{r}/(2\pi)^d$.

To reduce integrals of the type in eq. (2.12) to a small set of "master integrals", we employ symmetries following from exchanges of integration variables, as well as general partial integration identities for the spatial parts of the momentum integrations. The implementation of these identities follows the procedure outlined by Laporta [33], in analogy with reference [34]. We are lead both to very simple 1-loop recursion relations, such as

$$I(2i_1,0;0,0,0;k_1,1,0) = \frac{2k_1-2-d}{2k_1-2} I(2i_1-2,0;0,0,0;k_1-1,1,0)\,, \tag{2.13}$$

as well as well-known but less obvious 2-loop ones [35], like

$$I(0,0;0,0,0;1_\mathrm{b},1_\mathrm{b},1_\mathrm{b}) = 0\,, \tag{2.14}$$

where the subscripts refer to bosonic four-momenta.

After this reduction, only six master integrals remain:

$$I_\mathrm{b}(n) = \sum\!\!\!\!\!\!\!\int_{q_\mathrm{b}} \frac{1}{(q^2)^n}\,, \qquad I_\mathrm{f}(n) = \sum\!\!\!\!\!\!\!\int_{q_\mathrm{f}} \frac{1}{(q^2)^n}\,, \tag{2.15}$$

where $q_\mathrm{b}, q_\mathrm{f}$ refer to bosonic and fermionic Matsubara momenta, respectively, and $n = 1,2,3$. For a vanishing quark chemical potential, as we assume to be the case here, the fermionic integrals reduce further to the bosonic ones,

$$I_\mathrm{f}(n) = \left(2^{2n-d}-1\right) I_\mathrm{b}(n)\,, \tag{2.16}$$

leaving only three master integrals. They are known explicitly,

$$I_\mathrm{b}(n) = \frac{2\pi^{d/2}T^{1+d}}{(2\pi T)^{2n}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \zeta(2n-d)\,. \tag{2.17}$$

This expression is easily expanded in ϵ and, in the following, we need terms up to $\mathcal{O}(\epsilon)$. For completeness, the relevant expansions are shown in appendix A.

Writing now the Taylor-expanded bare 2-point function Π_T of equation (2.11) as

$$\begin{aligned}\Pi_\mathrm{T}(\mathbf{p}^2) &\equiv \Pi_\mathrm{T}(0) + \mathbf{p}^2\Pi'_\mathrm{T}(0) + \cdots \\ &\equiv \sum_{n=1}^{\infty} \Pi_{\mathrm{T}n}(0)(g_B^2)^n + \mathbf{p}^2 \sum_{n=1}^{\infty} \Pi'_{\mathrm{T}n}(0)(g_B^2)^n + \cdots\,,\end{aligned} \tag{2.18}$$

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where g_B is the bare gauge coupling, and correspondingly for $\Pi_{\rm E}$, our results read

$$
\begin{aligned}
\Pi_{\rm T1}(0) &= 0\,, && (2.19)\\
\Pi'_{\rm T1}(0) &= \frac{d-25}{6} C_A I_{\rm b}(2) + \frac{4}{3}\, T_F I_{\rm f}(2)\,, && (2.20)\\
\Pi_{\rm T2}(0) &= 0\,, && (2.21)\\
\Pi'_{\rm T2}(0) &= \frac{(d-3)(d-4)}{d(d-2)(d-5)(d-7)} \biggl\{ 2(4d^2-21d-7) C_A^2 I_{\rm b}^2(2) - \\
&\qquad - 8\Bigl[4C_F + (d^2-6d+1)C_A\Bigr] T_F I_{\rm b}(2) I_{\rm f}(2) - \\
&\qquad - \Bigl[(d^3-12d^2+39d-12)C_A - \\
&\qquad\quad - 2(d^3-12d^2+41d-14)C_F\Bigr] T_F I_{\rm f}^2(2) \biggr\} + \\
&\quad + \frac{(d-1)}{3d(d-7)} \biggl\{ (d^2-31d+144)\Bigl[4T_F I_{\rm f}(1) - (d-1)C_A I_{\rm b}(1)\Bigr] C_A I_{\rm b}(3) - \\
&\qquad - 8(d-1)(d-6) C_F T_F \Bigl[I_{\rm b}(1) - I_{\rm f}(1)\Bigr] I_{\rm f}(3) \biggr\}\,, && (2.22)\\
\Pi_{\rm E1}(0) &= -(d-1)\Bigl[4T_F I_{\rm f}(1) - (d-1)C_A I_{\rm b}(1)\Bigr]\,, && (2.23)\\
\Pi'_{\rm E1}(0) &= -\biggl[\frac{d^2-5d+28}{6} + (d-3)\xi\biggr] C_A I_{\rm b}(2) + \frac{2(d-1)}{3} T_F I_{\rm f}(2)\,, && (2.24)\\
\Pi_{\rm E2}(0) &= (d-1)(d-3)\biggl\{(1+\xi)\Bigl[4T_F I_{\rm f}(1) - (d-1)C_A I_{\rm b}(1)\Bigr] C_A I_{\rm b}(2) + \\
&\qquad + 4C_F T_F \Bigl[I_{\rm b}(1) - I_{\rm f}(1)\Bigr] I_{\rm f}(2)\biggr\}. && (2.25)
\end{aligned}
$$

We leave out the lengthy expression for $\Pi'_{\rm E2}(0)$, as it is not needed in the following.

The bare results need still to be renormalised. The bare gauge coupling is written as $g_B^2 = g^2(\bar\mu) Z_g$, where $g^2(\bar\mu)$ is the renormalised gauge coupling, $\bar\mu$ is an $\overline{\rm MS}$ scheme scale parameter introduced through $\mu^2 \equiv \bar\mu^2 e^{\gamma_{\rm E}}/4\pi$, and the combination $\mu^{-2\epsilon} g^2(\bar\mu)$ is dimensionless. Denoting

$$
\begin{aligned}
\beta_0 &\equiv \frac{-22C_A + 8T_F}{3}\,, && (2.26)\\
\beta_1 &\equiv \frac{-68C_A^2 + 40C_A T_F + 24 C_F T_F}{3}\,, && (2.27)
\end{aligned}
$$

the factor Z_g reads

$$
Z_g = 1 + \frac{1}{(4\pi)^2}\frac{\beta_0}{2\epsilon}\mu^{-2\epsilon} g^2(\bar\mu) + \frac{1}{(4\pi)^4}\biggl[\frac{\beta_1}{4\epsilon} + \frac{\beta_0^2}{4\epsilon^2}\biggr]\mu^{-4\epsilon} g^4(\bar\mu) + \mathcal{O}(g^6)\,, \tag{2.28}
$$

and the renormalised gauge coupling satisfies, in the limit $\epsilon \to 0$,

$$
\bar\mu \frac{{\rm d}}{{\rm d}\bar\mu} g^2(\bar\mu) = \frac{\beta_0}{(4\pi)^2} g^4(\bar\mu) + \frac{\beta_1}{(4\pi)^4} g^6(\bar\mu) + \mathcal{O}(g^8)\,. \tag{2.29}
$$

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To proceed, we first cross-check our results for Π_E against known expressions. After the fields B_0^a of EQCD are normalised to their canonical form (cf. eq. (2.6)), $(B_0^a B_0^a)_\mathrm{E}\, \mathbf{p}^2 \equiv (B_0^a B_0^a)_\mathrm{4d}\, \mathbf{p}^2[1+\Pi'_\mathrm{E1}(0)]/T$, we obtain for the matching coefficient m_E^2,

$$m_\mathrm{E}^2 = g_B^2\, \Pi_\mathrm{E1}(0) + g_B^4 \Big[\Pi_\mathrm{E2}(0) - \Pi'_\mathrm{E1}(0)\Pi_\mathrm{E1}(0)\Big] + \mathcal{O}(g_B^6)\,. \tag{2.30}$$

Inserting eqs. (2.23)–(2.25), the ξ-dependence duly cancels. Re-expanding also g_B^2 in terms of the renormalised gauge coupling, and writing then [10]

$$m_\mathrm{E}^2 \equiv T^2 \bigg\{ g^2(\bar\mu)\Big[\alpha_\mathrm{E4} + \alpha_\mathrm{E5}\epsilon\Big] + \frac{g^4(\bar\mu)}{(4\pi)^2}\Big[\alpha_\mathrm{E6} + \beta_\mathrm{E2}\epsilon\Big] + \mathcal{O}(g^6,\epsilon^2) \bigg\}, \tag{2.31}$$

we recover the known values of α_E4, α_E5 and α_E6 [10] (for original derivations, see reference [8] and references therein). We also obtain

$$\begin{aligned}
\beta_\mathrm{E2} = {} & \frac{1}{36} C_A^2 \bigg\{ 264 \ln^2\!\Big(\frac{\bar\mu e^{\gamma_\mathrm{E}}}{4\pi T}\Big) + \bigg[80 - 176\gamma_\mathrm{E} + 176\frac{\zeta'(-1)}{\zeta(-1)}\bigg] \ln\!\Big(\frac{\bar\mu e^{\gamma_\mathrm{E}}}{4\pi T}\Big) + \\
& \quad + 8 + 11\pi^2 - 88\gamma_\mathrm{E}^2 - 40\gamma_\mathrm{E} - 176\gamma_1 + 40\frac{\zeta'(-1)}{\zeta(-1)} \bigg\} + \\
& + C_F T_F \bigg\{ -8\ln\!\Big(\frac{\bar\mu e^{\gamma_\mathrm{E}}}{4\pi T}\Big) - 2 - \frac{20}{3}\ln 2 + 4\gamma_\mathrm{E} - 4\frac{\zeta'(-1)}{\zeta(-1)} \bigg\} + \\
& + \frac{1}{36} C_A T_F \bigg\{ 168 \ln^2\Big(\frac{\bar\mu e^{\gamma_\mathrm{E}}}{4\pi T}\Big) + \bigg[232 - 432\ln 2 - 112\gamma_\mathrm{E} + 112\frac{\zeta'(-1)}{\zeta(-1)}\bigg] \ln\!\Big(\frac{\bar\mu e^{\gamma_\mathrm{E}}}{4\pi T}\Big) + \\
& \quad + 28 + 7\pi^2 + 24\ln 2 - 64\ln^2 2 - 56\gamma_\mathrm{E}^2 - 72\gamma_\mathrm{E} + 128\gamma_\mathrm{E}\ln 2 - 112\gamma_1 + \\
& \quad + 72\frac{\zeta'(-1)}{\zeta(-1)} - 128\ln 2\frac{\zeta'(-1)}{\zeta(-1)} \bigg\} + \\
& + \frac{1}{9} T_F^2 \bigg\{ -24\ln^2\!\Big(\frac{\bar\mu e^{\gamma_\mathrm{E}}}{4\pi T}\Big) + \bigg[8 - 48\ln 2 + 16\gamma_\mathrm{E} - 16\frac{\zeta'(-1)}{\zeta(-1)}\bigg] \ln\!\Big(\frac{\bar\mu e^{\gamma_\mathrm{E}}}{4\pi T}\Big) + \\
& \quad + 4 - \pi^2 - 8\ln 2 + 16\ln^2 2 + 8\gamma_\mathrm{E}^2 - 8\gamma_\mathrm{E} + 32\gamma_\mathrm{E}\ln 2 + 16\gamma_1 + \\
& \quad + 8\frac{\zeta'(-1)}{\zeta(-1)} - 32\ln 2\frac{\zeta'(-1)}{\zeta(-1)} \bigg\}.
\end{aligned} \tag{2.32}$$

Here γ_1 is a Stieltjes constant, defined through the series $\zeta(s)=1/(s-1)+\sum_{n=0}^\infty \gamma_n(-1)^n(s-1)^n/n!$. (Note that the Euler gamma-constant is $\gamma_\mathrm{E}\equiv\gamma_0$.) The result in eq. (2.32), first obtained in reference [37] by employing the results of reference [8], contributes to the pressure of hot QCD at $\mathcal{O}(g^6T^4)$ [10]. We rewrite the expression here, since reference [37] employed an extremely compactified notation.

We then move to consider the transverse spatial part, $\Pi_\mathrm{T}(\mathbf{p}^2)$. According to eq. (2.9), this directly determines the effective gauge coupling:

$$g_\mathrm{E}^2 = T\Big\{ g_B^2 - g_B^4 \Pi'_{T1}(0) + g_B^6\Big[\Big(\Pi'_{T1}(0)\Big)^2 - \Pi'_{T2}(0)\Big] + \mathcal{O}(g_B^8)\Big\}. \tag{2.33}$$

Re-expanding again in terms of $g^2(\bar\mu)$, we parameterise the result (following ref. [10]) as

$$g_\mathrm{E}^2 \equiv T\bigg\{ g^2(\bar\mu) + \frac{g^4(\bar\mu)}{(4\pi)^2}\left[\alpha_\mathrm{E7} + \beta_\mathrm{E3}\epsilon + \mathcal{O}(\epsilon^2)\right] + \frac{g^6(\bar\mu)}{(4\pi)^4}\left[\gamma_\mathrm{E1} + \mathcal{O}(\epsilon)\right] + \mathcal{O}(g^8)\bigg\}. \tag{2.34}$$

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We recover the known expression [36, 12] for $\alpha_{\rm E7}$,

$$\alpha_{\rm E7} = -\beta_0 \ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right) + \frac{1}{3} C_A - \frac{16}{3} T_F \ln 2\,, \tag{2.35}$$

and obtain the new contributions

$$\begin{aligned}\beta_{\rm E3} &= \frac{1}{12} C_A \left[88 \ln^2\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right) + 8 \ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right) + 11\pi^2 - 88\gamma_{\rm E}^2 - 176\gamma_1 \right] - \\ &\quad - \frac{1}{3} T_F \left[8 \ln^2\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right) + 32 \ln 2 \ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right) + \pi^2 + 16 \ln^2 2 - 8\gamma_{\rm E}^2 - 16\gamma_1 \right], \end{aligned} \tag{2.36}$$

$$\begin{aligned}\gamma_{\rm E1} &= -\beta_1 \ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right) + \left[\beta_0 \ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right) - \frac{1}{3} C_A + \frac{16}{3} T_F \ln 2 \right]^2 - \\ &\quad - \frac{1}{18} \Big\{ C_A^2 \Big[-341 + 20\zeta(3) \Big] + 4 C_A T_F \Big[43 + 24 \ln 2 + 5\zeta(3) \Big] + \\ &\qquad + 6 C_F T_F \Big[23 + 80 \ln 2 - 14\zeta(3) \Big] \Big\}. \end{aligned} \tag{2.37}$$

The first one, $\beta_{\rm E3}$, constitutes again an $\mathcal{O}(g^6T^4)$ contribution to the pressure of hot QCD [10], while the latter one is the desired finite 2-loop correction to the effective gauge coupling.

3. Numerical evaluation

We wish to compare numerically the 1-loop and 2-loop expressions for $g_{\rm E}^2$, in the limit $\epsilon \to 0$. When carrying out such a comparison, it is important to specify the definitions of the $\Lambda_{\overline{\rm MS}}$-parameters. Following standard procedures, we solve eq. (2.29) exactly at 2-loop level, and define

$$\Lambda_{\overline{\rm MS}} \equiv \lim_{\bar\mu\to\infty} \bar\mu \left[b_0 g^2(\bar\mu) \right]^{-b_1/2b_0^2} \exp\left[-\frac{1}{2b_0 g^2(\bar\mu)} \right], \tag{3.1}$$

where $b_0 \equiv -\beta_0/2(4\pi)^2$, $b_1 \equiv -\beta_1/2(4\pi)^4$. For large $\bar\mu$ this leads to the usual behaviour

$$\frac{1}{g^2(\bar\mu)} \approx 2b_0 \ln\frac{\bar\mu}{\Lambda_{\overline{\rm MS}}} + \frac{b_1}{b_0} \ln\left(2 \ln\frac{\bar\mu}{\Lambda_{\overline{\rm MS}}} \right). \tag{3.2}$$

In the 1-loop case, we set $b_1 \equiv 0$ in eqs. (3.1), (3.2).

Through eqs. (2.34), (2.35), (2.37) and (3.2), $g_{\rm E}^2$ is a function $\bar\mu/T$ and $\bar\mu/\Lambda_{\overline{\rm MS}}$. The dependence on $\bar\mu$ is formally of higher order than the computation. Numerically, of course, there is non-vanishing dependence, as illustrated in figure 2.

As usual, one may choose some "optimisation" criterion which should lead to a reduced $\bar\mu$-dependence and thus reasonable convergence. We fix $\bar\mu_{\rm opt}$ to be the point where the 1-loop coupling $g_{\rm E}^2$ has vanishing slope ("principal of minimal sensitivity"), cf. figure 2, and vary then the scale in the range $\bar\mu = (0.5\ldots2.0)\times\bar\mu_{\rm opt}$ around this point. Results are shown in fig. 3. The $\bar\mu$-dependence indeed decreases significantly as we go to the 2-loop level. The numerical 2-loop value is some 20% smaller than the 1-loop value. It is comforting that the 2-loop value is on the side to which the "error band" of the 1-loop result points, even though it does not in general lie within that band.

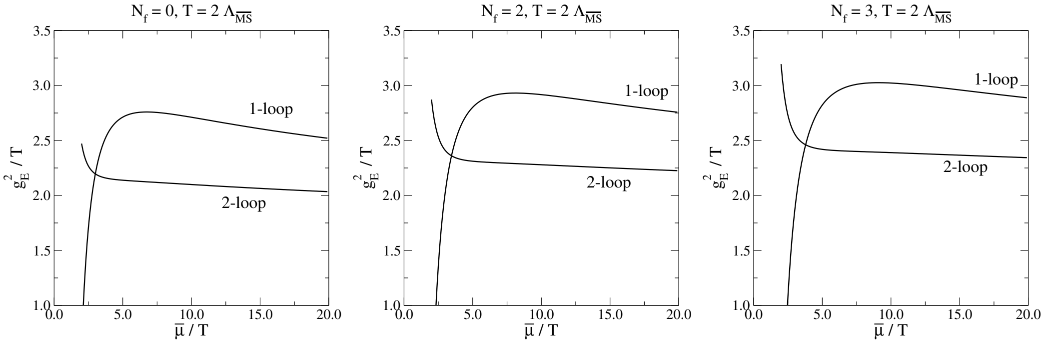

Figure 2: A comparison of 1-loop and 2-loop values for $g_{\rm E}^2/T$, as a function of $\bar\mu/T$, for a fixed $T/\Lambda_{\overline{\rm MS}} = 2.0$ and $N_{\rm f} = 0, 2, 3$.

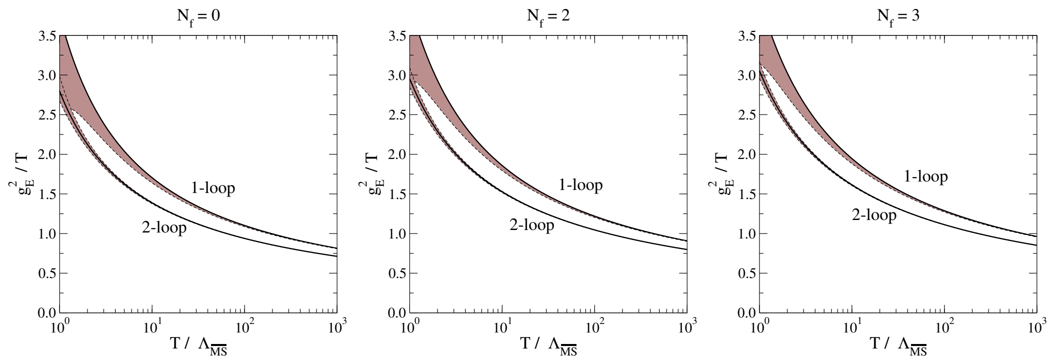

Figure 3: The 1-loop and 2-loop values for $g_{\rm E}^2/T$, as a function of $T/\Lambda_{\overline{\rm MS}}$ (solid lines). For each T the scale $\bar\mu$ has been fixed to the "principal of minimal sensitivity" point $\bar\mu_{\rm opt}$ following from the 1-loop expression, and varied then in the range $\bar\mu = (0.5 \ldots 2.0) \times \bar\mu_{\rm opt}$ (the grey bands).

4. Spatial string tension

The computations in the previous sections can be given a "phenomenological" application, by considering lattice measurements of the so-called spatial string tension. The spatial string tension is obtained from a rectangular Wilson loop $W_s(R_1, R_2)$ in the (x_1, x_2)-plane, of size $R_1 \times R_2$. The potential $V_s(R_1)$ is defined through

$$V_s(R_1) = - \lim_{R_2\to\infty} \frac{1}{R_2} \ln W_s(R_1, R_2)\,, \tag{4.1}$$

and the spatial string tension σ_s from the asymptotic behaviour of the potential,

$$\sigma_s \equiv \lim_{R_1\to\infty} \frac{V_s(R_1)}{R_1}\,. \tag{4.2}$$

Since σ_s has the dimensionality GeV^2, it is often expressed [22] as the combination

$$\frac{\sqrt{\sigma_s}}{T} = \phi\Big(\frac{T}{T_{\rm c}}\Big)\,, \tag{4.3}$$

where ϕ is a (decreasing) dimensionless function, and $T_{\rm c}$ is the critical temperature of the deconfinement phase transition.

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We now turn to how the result for $g_{\rm E}^2$ that we have obtained in this paper, combined with other ingredients, allow us to obtain an independent prediction for the spatial string tension.

4.1 Three-dimensional prediction

The very same observable as in eq. (4.1), exists also in 3d SU(3) gauge theory, or "Magnetostatic QCD" (MQCD). Since the gauge coupling $g_{\rm M}^2$ of MQCD is dimensionful, σ_s must have the form $\sigma_s = c \times g_{\rm M}^4$, where c is a numerical proportionality constant. It has been determined with lattice Monte Carlo methods most recently in reference [24] where, after the continuum extrapolation, it was expressed as

$$\frac{\sqrt{\sigma_s}}{g_{\rm M}^2} = 0.553(1)\,. \tag{4.4}$$

In order to compare eqs. (4.3), (4.4), we need a relation between T and $g_{\rm M}^2$. In the previous section, we obtained a relation between T and $g_{\rm E}^2$. The relation between $g_{\rm E}^2$ and $g_{\rm M}^2$ is also known, up to 2-loop order [25]:[2]

$$g_{\rm M}^2 = g_{\rm E}^2\left[1 - \frac{1}{48}\frac{g_{\rm E}^2 C_A}{\pi m_{\rm E}} - \frac{17}{4608}\left(\frac{g_{\rm E}^2 C_A}{\pi m_{\rm E}}\right)^2\right], \tag{4.5}$$

where the 1-loop part was determined already in reference [26].

It is worth stressing that the corrections in eq. (4.5) are in practice extremely small, even for values of $m_{\rm E}/g_{\rm E}^2$ corresponding to temperatures very close to the critical one. (For $N_{\rm c} = 3$ and $N_{\rm f} = 0$, $(m_{\rm E}/g_{\rm E}^2)^2 \approx 0.32\,\log_{10}(T/\Lambda_{\overline{\rm MS}}) + 0.29$.) This seems by no means obvious *a priori*, given the observed slow convergence in the case of the vacuum energy density of EQCD [10]. In view of this fact, however, we can safely ignore all higher loop corrections in eq. (4.5).

Another source of errors in going from EQCD to MQCD are the higher order operators that have been truncated from the action of MQCD. As discussed in reference [10], they are expected to contribute at the relative order $\mathcal{O}(g_{\rm E}^6/m_{\rm E}^3)$, i.e. at the same order that 3-loop corrections enter eq. (4.5). From this consideration, one might expect them to again be numerically negligible. In principle one could avoid this assumption, however: the ratio $\sqrt{\sigma_s}/g_{\rm E}^2$ has been estimated in reference [17] through direct numerical simulations in EQCD. Unfortunately the statistical and particularly the systematic errors appear to be non-vanishing (no continuum extrapolation was carried out for this quantity), so that we prefer to follow the line starting from eq. (4.4) in the following. Nevertheless it would be interesting to learn more about the importance of the higher order operators.

Now, as we know $g_{\rm E}^2/T$ as a function of $T/\Lambda_{\overline{\rm MS}}$ from fig. 3, eqs. (4.4) and (4.5) allow us to obtain $\sqrt{\sigma_s}/T$ as a function of the same variable. In order to compare with eq. (4.3), however, we still need to relate $\Lambda_{\overline{\rm MS}}$ to $T_{\rm c}$. This problem has also been addressed with 4d lattice simulations, as we review in section 4.2.

[2] The 2-loop correction $\delta g_{\rm M}^2/g_{\rm E}^2 = -g_{\rm E}^2 C_A[2(C_A C_F + 1)\lambda_{\rm E}^{(1)} + (6C_F - C_A)\lambda_{\rm E}^{(2)}]/384(\pi m_{\rm E})^2$ was ignored in reference [25], as it is of higher order according to 4d power counting and numerically insignificant.

4.2 Critical temperature in "perturbative units"

The determination of $T_c/\Lambda_{\overline{\rm MS}}$ is a classic problem in lattice QCD. Two main lines have been followed, one going via the zero temperature string tension $\sqrt{\sigma}$, the other via the Sommer scale r_0 [38].

Values obtained for $T_c/\sqrt{\sigma}$ by various lattice collaborations are summarised in reference [39], Table 7. Traditionally the values were around $T_c/\sqrt{\sigma} = 0.630(5)$ [40], but reference [39] argues in favour of a slightly larger number in the continuum limit. Indeed the most precise estimate appears to come from reference [41], where $T_c/\sqrt{\sigma} = 0.646(3)$ is cited. Combining with $\Lambda_{\overline{\rm MS}}/\sqrt{\sigma} = 0.555(19)$ from reference [42], we are lead to

$$\frac{T_c}{\Lambda_{\overline{\rm MS}}} = 1.16(4)\,. \tag{4.6}$$

The error is dominated by the one in $\Lambda_{\overline{\rm MS}}/\sqrt{\sigma}$.

A value for r_0T_c, on the other hand, has been obtained in reference [43]: $r_0T_c = 0.7498(50)$. Combining with $r_0\Lambda_{\overline{\rm MS}} = 0.602(48)$ from reference [44] (the value $r_0\Lambda_{\overline{\rm MS}} = 0.586(48)$ from a few lines below eq. (4.11) in reference [45] is well within error bars), one obtains

$$\frac{T_c}{\Lambda_{\overline{\rm MS}}} = 1.25(10)\,. \tag{4.7}$$

This is consistent, within statistical errors, with eq. (4.6), if favouring a slightly larger central value. Again the error is dominated by the zero-temperature part, $r_0\Lambda_{\overline{\rm MS}}$ in this case. In general it might be expected, though, that systematic uncertainties are better under control in the extraction of r_0 than of $\sqrt{\sigma}$, since the static potential needs to be computed only up to intermediate distances.

Apart from going through $\sqrt{\sigma}$ and r_0, there is also a third possibility [46]. It is based on directly determining a (lattice) Λ-parameter from the scaling of a suitably defined renormalised gauge coupling at the critical point, and converting at the end to the $\overline{\rm MS}$ scheme. The value obtained is

$$\frac{T_c}{\Lambda_{\overline{\rm MS}}} = 1.15(5)\,, \tag{4.8}$$

consistent with eqs. (4.6) and (4.7).

To be conservative, we will consider the interval $T_c/\Lambda_{\overline{\rm MS}} = 1.10\ldots1.35$ in the following, encompassing the central values as well as the error bars of eqs. (4.6)–(4.8).

4.3 Four-dimensional measurement

The spatial string tension of 4d pure SU(3) gauge theory at temperatures above the critical one, as a function of T/T_c, has been measured at $N_\tau = 8$ in reference [22] (cf. figure 11). There are, of course, systematic uncertainties, both from the lack of a continuum extrapolation as well as from how the string tension is extracted by fitting to the large-distance behaviour of the static potential. Nevertheless, we expect that the results are in the right ballpark.

Given the considerations in sections 4.1, 4.2, we can thus compare the 3d and the 4d determinations of $\sqrt{\sigma_s}/T$. The result is shown in fig. 4, where $T/\sqrt{\sigma_s}$ is plotted. We observe

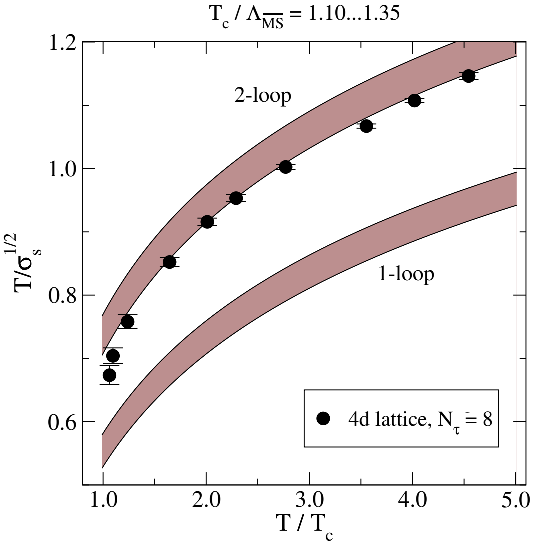

Figure 4: We compare 4d lattice data for the spatial string tension, taken from reference [22], with expressions obtained by combining 1-loop and 2-loop results for $g_{\rm E}^2$ together with eq. (4.5) and the non-perturbative value of the string tension of 3d SU(3) gauge theory, eq. (4.4). The upper edges of the bands correspond to $T_{\rm c}/\Lambda_{\overline{\rm MS}} = 1.35$, the lower edges to $T_{\rm c}/\Lambda_{\overline{\rm MS}} = 1.10$.

a significant discrepancy at 1-loop level (as most recently pointed out in reference [23]), but a remarkable agreement once we go to 2-loop level. It is also noteworthy that the functional form of the 2-loop curve appears to match the behaviour of the lattice data down to low temperatures.

5. Conclusions

The main purpose of this paper has been the analytic computation of the 2-loop effective gauge coupling of QCD at finite temperatures, defined as a matching coefficient appearing in the dimensionally reduced effective theory, EQCD.[3] The result is given in eqs. (2.34)–(2.37). We have also determined a new contribution of order $\mathcal{O}(g^6T^4)$ to the pressure of hot QCD; the information is contained in eq. (2.36), and how it enters the pressure is explained in reference [10].

The 2-loop correction we find is numerically substantial, some 20% of the 1-loop expression. This indicates that while perturbation theory is in principle still under control, if restricted to the parametrically hard modes $p \sim 2\pi T$ only, it is important to push it to a sufficiently high order, in order to obtain precise results.

Our expression for the effective gauge coupling has a direct "phenomenological" application, in that it allows for a parameter-free comparison of 3d MQCD and 4d full theory results for an observable called the spatial string tension. We find that the 2-loop correction computed here improves the match between the two results quite significantly, down

[3]Other "effective gauge couplings" can of course also be defined; for a recent review, see reference [47]. The difference is that in these cases all momentum scales influence the effective gauge coupling, so that perturbation theory cannot be reliably applied for its computation.

to temperatures very close to the critical one. A small discrepancy still remains but, given that no continuum extrapolation was taken in 4d lattice simulations, that the extraction of the spatial string tension may involve systematic uncertainties due to large subleading terms in the r-dependence of the spatial static potential $V_s(r)$ [48], and that there also has to be some room for residual 3-loop corrections, as well as improvements in the matching between EQCD and MQCD, we do not consider this discrepancy to be worrying. We do believe that the discrepancy can be decreased by improving systematically on the various ingredients that enter the comparison.

These conclusions support a picture of thermal QCD according to which the *parametrically* "hard" scales, $p \sim 2\pi T$, can be treated perturbatively, almost as soon as we are in the deconfined phase, will the *parametrically* "soft" scales, $p \sim gT, g^2T$, require in general a non-perturbative analysis within one of the effective theories describing their dynamics. For the observable we considered here, in fact, even the colour-electric scale $p \sim gT$ could be integrated out perturbatively, but it is known that this is in general not the case. We should like to stress that this conclusion is rather non-trivial, as there *numerically* is little hierarchy between the scales $2\pi T, gT, g^2T$ at the realistic temperatures that we have been considering.

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A. Expansions for master integrals

Using the notation introduced in the text, the master integrals of eq. (2.17) read, up to $\mathcal{O}(\epsilon)$:

$$I_{\rm b}(1)=\mu^{-2\epsilon}\frac{T^2}{12}\left\{1+\epsilon\left[2\ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right)+2-2\gamma_{\rm E}+2\frac{\zeta'(-1)}{\zeta(-1)}\right]\right\}, \tag{A.1}$$

$$I_{\rm b}(2)=\mu^{-2\epsilon}\frac{1}{(4\pi)^2}\left\{\frac{1}{\epsilon}+2\ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right)+\epsilon\left[2\ln^2\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right)+\frac{\pi^2}{4}-2\gamma_{\rm E}^2-4\gamma_1\right]\right\}, \tag{A.2}$$

$$I_{\rm b}(3)=\mu^{-2\epsilon}\frac{\zeta(3)}{128\pi^4T^2}\left\{1+\epsilon\left[2\ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\right)+2-2\gamma_{\rm E}+2\frac{\zeta'(3)}{\zeta(3)}\right]\right\}, \tag{A.3}$$

$$I_{\rm f}(1)=\mu^{-2\epsilon}\left(-\frac{T^2}{24}\right)\left\{1+\epsilon\left[2\ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{\pi T}\right)+2-6\ln 2-2\gamma_{\rm E}+2\frac{\zeta'(-1)}{\zeta(-1)}\right]\right\}, \tag{A.4}$$

$$I_{\rm f}(2)=\mu^{-2\epsilon}\frac{1}{(4\pi)^2}\left\{\frac{1}{\epsilon}+2\ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{\pi T}\right)+\epsilon\left[2\ln^2\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{\pi T}\right)+\frac{\pi^2}{4}-4\ln^2 2-2\gamma_{\rm E}^2-4\gamma_1\right]\right\}, \tag{A.5}$$

$$I_{\rm f}(3)=\mu^{-2\epsilon}\frac{7\zeta(3)}{128\pi^4T^2}\left\{1+\epsilon\left[2\ln\left(\frac{\bar\mu e^{\gamma_{\rm E}}}{\pi T}\right)+2-\frac{12}{7}\ln 2-2\gamma_{\rm E}+2\frac{\zeta'(3)}{\zeta(3)}\right]\right\}. \tag{A.6}$$

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